

# Convex Functions and Spacetime Geometry

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## Abstract

Convexity and convex functions play an important role in theoretical physics. To initiate a study of the possible uses of convex functions in General Relativity, we discuss the consequences of a spacetime  $(M, g_{\mu\nu})$  or an initial data set  $(\Sigma, h_{ij}, K_{ij})$  admitting a suitably defined convex function. We show how the existence of a convex function on a spacetime places restrictions on the properties of the spacetime geometry.

## 1 Introduction

Convexity and convex functions play an important role in theoretical physics. For example, Gibbs's approach to thermodynamics [G] is based on the idea that the free energy should be a convex function. A closely related concept is that of a convex cone which also has numerous applications to physics. Perhaps the most familiar example is the lightcone of Minkowski spacetime. Equally important is the convex cone of mixed states of density matrices in quantum mechanics. Convexity and convex functions also have important applications to geometry, including Riemannian geometry [U]. It is surprising therefore that, to our knowledge, that techniques making use of convexity and convex functions have played no great role in General Relativity. The purpose of this paper is to initiate a study of the possible uses of such techniques. As we shall see, the existence of a convex function, suitably defined, on a spacetime places important restrictions on the properties of the spacetime. For example it can contain no closed spacelike geodesics. We shall treat Riemannian as well as Lorentzian manifolds with an eye to applications in black hole physics, cosmology and quantum gravity. The detailed results and examples will be four-dimensional but extensions to other dimensions are immediate.

One way of viewing the ideas of this paper is in terms of a sort of duality between paths and particles on the one hand and functions and waves on the other. Mathematically the duality corresponds to interchanging range and domain. A curve  $x(\lambda)$  is a map  $x : \mathbb{R} \rightarrow M$  while a function  $f(x)$  is a map  $f : M \rightarrow \mathbb{R}$ . A path arises by considering invariance under diffeomorphisms of the domain (i.e. of the world volume) and special paths, for example geodesics have action functionals which are reparametrization invariant. Much effort, physical and mathematical has been expended on exploring the global properties of spacetimes using geodesics. Indeed there is a natural notion of convexity based on geodesics. Often a congruence of geodesics is used.

On the dual side, one might consider properties which are invariant under diffeomorphisms  $f(x) \rightarrow g(f(x))$  of the range or target space. That is one may explore the global properties of spacetime using

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the foliations provided by the level sets of a suitable function. The analogue of the action functional for geodesics is one like

$$\int_M \sqrt{\epsilon \nabla_\mu f \nabla^\mu f}, \quad (1)$$

where  $\epsilon = \pm$ , depending upon whether  $\nabla_\mu f$  is spacelike or timelike, and which is invariant under reparametrizations of the range. The level sets of the solutions of the Euler-Lagrange equations are then minimal surfaces. One may also consider foliations by totally umbilic surfaces or by “trace  $K$  equal constant” foliations, and this is often done in numerical relativity. The case of a convex function then corresponds to a foliation by totally expanding hypersurfaces, that is, hypersurfaces with positive definite second fundamental form.

Another way of thinking about spacetime is in terms of the causal structure provided by the Lorentzian metric. However what we shall call a spacetime convex function has a Hessian with Lorentzian signature which also defines a causal structure, the light cone of which always lies inside the usual spacetime light-cone. The spacetimes admitting spacetime convex functions have particular types of causal structures.

The organisation of the paper is as follows. In the next section, we consider convex functions on Riemannian manifolds, which are regarded as spacelike submanifolds embedded in a spacetime or as Euclidean solutions of Einstein’s equation. We see that the existence of convex functions of strictly or uniformly convex type is incompatible with the existence of closed minimal submanifolds. We also discuss the relation between the convex functions and Killing vector fields. In section 3, we give two definitions for convex functions on Lorentzian manifolds: a *classical convex function* and a *spacetime convex function*. In particular, the latter defines a causal structure on a spacetime and has many important applications. We discuss the connection between spacetime convex functions and homothetic Killing vector fields. We show that, if a spacetime admits a spacetime convex function, then such a spacetime cannot have a marginally inner and outer trapped surface. In section 4, we examine in what case a cosmological spacetime admits a spacetime convex function. We also examine the existence of a spacetime convex function on specific spacetimes, de-Sitter, anti-de-Sitter, and black hole spacetimes, in the subsequent sections 5, 6, and 7, respectively. In section 8, we discuss the level sets of convex functions and foliations. Then, in section 9, we consider constant mean curvature foliations, barriers, and convex functions. Section 10 is devoted to a summary.

## 2 Convex functions on Riemannian manifolds

### 2.1 Definitions and a standard example

We shall briefly recapitulate standard definitions of convex functions on an  $n$ -dimensional Riemannian manifold  $(\Sigma, h_{ij})$ . What follows immediately is applicable to any Riemannian manifold of dimension  $n$  including one obtained as a spacelike submanifold of an  $(n + 1)$ - dimensional spacetime.

**Definition 1 Convex function:**

A  $C^\infty$  function  $\tilde{f} : \Sigma \rightarrow \mathbb{R}$  is said to be (strictly) convex if the Hessian is positive semi-definite or positive definite, respectively

$$(i) \quad D_i D_j \tilde{f} \geq 0, \quad (2)$$

$$(ii) \quad D_i D_j \tilde{f} > 0, \quad (3)$$

where  $D_i$  is the Levi-Civita connection of the positive definite metric  $h_{ij}$ .

Properties of convex functions on Riemannian manifolds have been extensively studied [U].

**Convex Cones**

The set of convex functions on a Riemannian manifold  $(\Sigma, h)$  forms a *convex cone*  $C_\Sigma$ , that is, an

open subset of a vector space which is a cone:  $\tilde{f} \in C_\Sigma \Rightarrow \lambda \tilde{f} \in C_\Sigma, \forall \lambda \in \mathbb{R}_+$  and which is convex:  $\tilde{f}, \tilde{f}' \in C_\Sigma \Rightarrow \lambda \tilde{f} + (1 - \lambda) \tilde{f}' \in C_\Sigma, \forall \lambda \in (0, 1)$ .

Suppose  $\Sigma$  is a product  $\Sigma = \Sigma_1 \times \Sigma_2$  with metric direct sum  $h = h_1 \oplus h_2$ . Then it follows that  $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 \in C_\Sigma$ , for any  $\tilde{f}_1 \in C_{\Sigma_1}, \tilde{f}_2 \in C_{\Sigma_2}$ . This defines a direct sum of convex cones,

$$C_\Sigma = C_{\Sigma_1} \oplus C_{\Sigma_2} . \quad (4)$$

**Definition 2 Uniformly convex function:**

A smooth function  $\tilde{f} : \Sigma \rightarrow \mathbb{R}$  is said to be uniformly convex if there is a positive constant  $\tilde{c}$  such that

$$(iii) \quad D_i D_j \tilde{f} \geq \tilde{c} h_{ij}, \quad (5)$$

i.e.,  $\forall V \in T\Sigma, V^i V^j D_i D_j \tilde{f} \geq \tilde{c} h_{ij} V^i V^j$ .

The standard example is an  $\mathbb{E}^n$  with coordinates  $\{x^i\}$  and

$$\tilde{f} = \frac{1}{2} \{ (x^1)^2 + (x^2)^2 + \cdots + (x^n)^2 \} . \quad (6)$$

Note that the vector field  $\tilde{D}^i := h^{ij} D_j \tilde{f}$  satisfies

$$\mathcal{L}_{\tilde{D}} h_{ij} \geq 2\tilde{c} h_{ij} . \quad (7)$$

If inequality is replaced by equality we have a homothetic conformal Killing vector field. In the example above,

$$\tilde{D} = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + \cdots + x^n \frac{\partial}{\partial x^n} , \quad (8)$$

and  $\tilde{D}$  is the standard dilatation vector field. Thus,  $\Sigma$  admits a homothety.

## 2.2 Convex functions and submanifolds

Suppose  $(\Sigma, h_{ij})$  admits a convex function of the type (ii), or (iii). Then we deduce the following propositions.

**Proposition 1**  $\Sigma$  cannot be closed.

*Proof.* Otherwise by contraction (ii) or (iii) with  $h^{ij}$  it would admit a non-constant subharmonic function.  $\square$

**Proposition 2**  $(\Sigma, h_{ij})$  admits no closed geodesic curves.

*Proof.* Along any such a geodesic curve, we have

$$\frac{d^2 \tilde{f}}{ds^2} \geq \tilde{c} > 0 , \quad (9)$$

with  $s$  being an affine parameter of the geodesic curve. This contradicts to the closedness of the geodesic curve.  $\square$

**Proposition 3**  $(\Sigma, h_{ij})$  admits no closed minimal submanifold  $S \subset \Sigma$ .

*Proof.* Let  $k_{pq}$  ( $p, q = 1, 2, \dots, n-1$ ),  $H_{pq}$ ,  ${}^{(k)}\nabla_p$ , and  $D_i$  be the induced metric on  $S$ , the second fundamental form of  $S$ , the Levi-Civita connection of  $k_{pq}$ , and the projected Levi-Civita connection of  $h_{ij}$ , respectively. Then we have

$$D_p D_q \tilde{f} = {}^{(k)}\nabla_p {}^{(k)}\nabla_q \tilde{f} - H_{pq} \frac{\partial \tilde{f}}{\partial n}, \quad (10)$$

where  $\partial \tilde{f} / \partial n = n^i \partial_i \tilde{f}$  with  $n^i$  being the unit normal to  $S$ . Contracting with  $h^{ij}$  and using the fact that  $S$  is minimal, i.e.,  $k^{pq} H_{pq} = 0$ , we find that  $S$  admits a subharmonic function, i.e.,

$${}^{(k)}\nabla^2 \tilde{f} > 0. \quad (11)$$

However if  $S$  is closed this gives a contradiction.  $\square$

As a simple example, which also illustrates some of the subtleties involved, consider the following metric,

$$h_{ij} dx^i dx^j = \left(1 + \frac{M}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega^2), \quad M > 0, \rho > 0. \quad (12)$$

This is the Schwarzschild initial data in isotropic coordinates. The two-surface  $S$  defined as  $\rho = M/2$  ( $r = 2M$ ) is totally geodesic ( $H_{pq} = 0$ ) and hence minimal ( $k^{pq} H_{pq} = 0$ ). Thus  $\Sigma$  admits no convex function. Note that if we are concerned only with the region  $\rho > M/2$  (outside the trapped region) the restriction of the function

$$\tilde{f} = \frac{1}{2} \left(1 + \frac{M}{2\rho}\right)^4 \rho^2 \quad (13)$$

is a strictly convex function. However it ceases to be *strictly* convex on the totally geodesic submanifold at  $\rho = M/2$ . Restricted to  $\rho = M/2$ , the function is a constant, and hence not (strictly) subharmonic. Because  $\rho = M/2$  is closed and totally geodesic it contains closed geodesics which are also closed geodesics of the ambient manifold. In fact every such geodesic is a great circle on the 2-sphere  $\rho = M/2$ . Again this shows that the complete initial data set admits no strictly convex function.

From a spacetime view point,  $(\Sigma, h_{ij})$  in the example above is a time-symmetric hypersurface and the two-surface  $S$  at  $\rho = M/2$  is a marginally inner and outer trapped surface in the maximally extended Schwarzschild solution. As an application to time symmetric initial data sets, consider  $(\Sigma, h_{ij}, K_{ij})$  with the vanishing second fundamental form  $K_{ij} = 0$ . Then  $S \subset \Sigma$  would have been a marginally inner and outer trapped surface. We can observe that if  $\Sigma$  admits a convex function  $\tilde{f}$ , then no such apparent horizons can exist. This will be discussed further in the next section.

### Level sets

Level sets of convex functions have positive extrinsic curvature, in other words, the second fundamental form is positive definite. We will discuss this further in Section 8.

## 2.3 Convex functions and Killing vector fields

Killing vector fields are important in General Relativity and it is interesting therefore that there are connections with convex functions (see [U] and references there-in).

**Proposition 4** *Let  $\xi^i$  be a Killing vector field on  $(\Sigma, h_{ij})$  and define*

$$\tilde{f} := \frac{1}{2} \xi_i \xi^i. \quad (14)$$

*If the sectional curvature  $R_{iljm} \xi^l \xi^m$  of  $(\Sigma, h_{ij})$  is non positive, then  $\tilde{f}$  is (but not necessarily strictly) convex.*

*Proof.* From (14) and the Killing equation  $D_i \xi_j + D_j \xi_i = 0$ , we obtain

$$D_i D_j \tilde{f} = D_i \xi^m D_j \xi_m - R_{iljm} \xi^l \xi^m . \quad (15)$$

Hence  $R_{iljm} \xi^l \xi^m \leq 0$  yields  $D_i D_j \tilde{f} \geq 0$ .  $\square$

We shall give a few examples for which the function (14) becomes a convex function. A simple example is a Killing vector field  $\xi^i \partial_i = \partial_\tau$  on Euclidean Rindler space

$$h_{ij} dx^i dx^j = \alpha \chi^2 d\tau^2 + d\chi^2 , \quad (16)$$

with a positive constant  $\alpha$ , for which we have

$$D_i D_j \tilde{f} = \alpha h_{ij} > \tilde{c} h_{ij} . \quad (17)$$

Our next example is a Killing vector field,  $\xi^i \partial_i = \partial_\tau$ , on an Euclidean anti-de-Sitter space or a hyperbolic space  $\mathbb{H}^n$ ,

$$h_{ij} dx^i dx^j = (1 + r^2) d\tau^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_{(n-2)}^2 . \quad (18)$$

The function  $\frac{1}{2}(1 + r^2)$  and hence  $\frac{1}{2}r^2$  are convex.

The function achieves its minimum at  $r = 0$ , which is a geodesic. Moreover if one identifies  $\tau$  (with any arbitrary period) it becomes a closed geodesic, along which  $\tilde{f}$  is constant. Thus  $\tilde{f}$  is not a *strictly* convex function.

Another example is Euclidean Schwarzschild spacetime

$$h_{ij} dx^i dx^j = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \Omega_{pq} dz^p dz^q . \quad (19)$$

If  $r > 2M$ , then the function (14) with a Killing vector field  $\xi^i \partial_i = \partial_\tau$  is convex. However this function cannot be extended to the complete manifold with  $r \geq 2M$ ,  $0 \leq \tau \leq 8\pi M$  as a *strictly* convex function because the two-dimensional fixed point set  $r = 2M$  is compact (indeed it is topologically and geometrically a 2-sphere) and totally geodesic. This means it is totally geodesic and hence a closed minimal submanifold. As we have seen this is incompatible with the existence of a strictly convex function.

In general, a two-dimensional fixed point set of a Killing vector field in a Riemannian 4-manifold is called a *bolt* [GH]. We obtain the following

**Proposition 5** *A Riemannian 4-manifold admitting a strictly convex function can admit no Killing field with a closed bolt.*

## 3 Spacetime convex functions

### 3.1 Definitions and a canonical example

We shall define a convex function on Lorentzian manifolds. We have two different definitions of convex function on an  $(n + 1)$ -dimensional spacetime  $(M, g_{\mu\nu})$  as follows.

**Definition 3 Classical definition:**

A  $C^\infty$  function  $f : M \rightarrow \mathbb{R}$  is said to be classically convex if the Hessian is positive, i.e.,

$$(i) \quad \nabla_\mu \nabla_\nu f > 0 , \quad (20)$$

where  $\nabla_\mu$  is the Levi-Civita connection of  $g_{\mu\nu}$ .

**Definition 4 Spacetime definition:**

A smooth function  $f : M \rightarrow \mathbb{R}$  is called a spacetime convex function if the Hessian  $\nabla_\mu \nabla_\nu f$  has Lorentzian signature and satisfies the condition,

$$(ii) \quad \nabla_\mu \nabla_\nu f \geq c g_{\mu\nu} \quad (21)$$

with a positive constant  $c$ , i.e.,  $V^\mu V^\nu \nabla_\mu \nabla_\nu f \geq c g_{\mu\nu} V^\mu V^\nu$ , for  $\forall V^\mu \in TM$ .

The geometrical interpretation of the spacetime definition is that the forward light cone  $\mathcal{C}_f$  defined by the Hessian  $f_{\mu\nu} := \nabla_\mu \nabla_\nu f$  lies (strictly) inside the light cone  $\mathcal{C}_g$  defined by the spacetime metric  $g_{\mu\nu}$  as depicted in the figure 1. This means, in some intuitive sense, that the condition (ii) prevents the “collapse” of the light cone defined by  $g_{\mu\nu}$ .

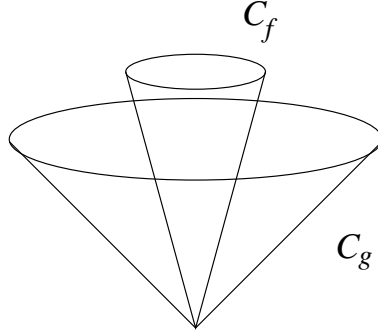


Figure 1: The light cones  $\mathcal{C}_f$  and  $\mathcal{C}_g$ .

One of the simplest examples of a spacetime convex function is the canonical one,

$$f = \frac{1}{2} (x^i x^i - \alpha t^2) \quad , \quad (t, x^i) \in \mathbb{E}^{n,1} \quad , \quad (22)$$

where  $\alpha$  is a constant such that  $0 < \alpha \leq 1$ .

The Hessian is given by

$$f_{\mu\nu} = \eta_{\mu\nu} + (1 - \alpha) t_\mu t_\nu \quad (23)$$

where  $\eta_{\mu\nu}$  is the Minkowski metric and  $t_\mu := \partial_\mu t$ , therefore  $f_{\mu\nu}$  actually has Lorentzian signature and can define a light cone  $\mathcal{C}_f$ . In the particular case  $\alpha = 1$ ,  $f_{\mu\nu} = \eta_{\mu\nu}$  and  $c$  must be unit.

Note that our conventions mean that, considered as a function of time,  $f$  in (22) is what is conventionally called a “concave function of time.” However it is a conventional convex function of  $x^i$ .

In the canonical example (22),  $\mathbb{E}^{n,1} = (\mathbb{R}, -dt^2) \times \mathbb{E}^n$  and  $f$  may be regarded as a direct sum of a concave function  $-\alpha t^2/2$  on  $(\mathbb{R}, -dt^2)$  and a convex function  $x^i x^i/2$  on  $\mathbb{E}^n$ . In general, when a spacetime  $(M, g)$  splits isometrically as a product  $(\mathbb{R} \times \Sigma, -dt^2 \oplus h)$ , one can construct a spacetime convex function by summing concave functions of time  $t \in \mathbb{R}$  and convex functions on  $(\Sigma, h)$ . However a direct sum of a convex cone  $C_{\mathbb{R}}$  which consists of concave functions of time  $t \in \mathbb{R}$  and a convex cone  $C_\Sigma$  consisting of convex functions on  $\Sigma$ , i.e.,

$$C_{\mathbb{R}} \oplus C_\Sigma \quad , \quad (24)$$

is not a convex cone  $C_M$  which consists of spacetime convex functions on  $(M, g)$ . This can be seen by considering the range of  $\alpha$  in (22). (i) If we take  $\alpha < 0$ ,  $f_{\mu\nu}$  becomes positive definite, hence  $f$  is classically convex, and (ii) if  $\alpha = 0$ ,  $f_{\mu\nu}$  becomes positive semi-definite ( $t^\mu t^\nu f_{\mu\nu} = 0$ ), hence  $f$  is not strictly classically convex, and (iii) if  $\alpha > 1$ ,  $f$  is not a convex function either in the classical or in the spacetime sense.

### 3.2 Hypersurface orthogonal homothetic Killing vector field

**Proposition 6** *If  $f$  is a spacetime convex function and  $\nabla^\mu f$  is a conformal Killing vector field,  $\nabla^\mu f$  must be homothetic.*

*Proof.* Let  $t^\mu$  and  $s^\mu$  be any unit timelike and spacelike vector, respectively. Then, by definition of the spacetime convex function, we have  $t^\mu t^\nu \nabla_\mu \nabla_\nu f \geq -c$ , and  $s^\mu s^\nu \nabla_\mu \nabla_\nu f \geq c$ . On the other hand, when  $\nabla_\mu f$  is a conformal Killing vector field, i.e.,  $\nabla_\mu \nabla_\nu f = \phi(x)g_{\mu\nu}$ , it follows that  $t^\mu t^\nu \nabla_\mu \nabla_\nu f = -\phi$ , and  $s^\mu s^\nu \nabla_\mu \nabla_\nu f = \phi$ . Therefore  $\phi(x) = c$ .  $\square$

It is easy to see that, if  $f$  is classically convex,  $\nabla_\mu f$  cannot be a Killing vector field.

Suppose that we have a hypersurface orthogonal homothetic vector field  $D$ . An example is  $D = x^\mu \frac{\partial}{\partial x^\mu}$  in flat Minkowski spacetime. In general there will be regions where  $D$  is timelike and regions where it is spacelike, separated by a null hypersurface on which  $D$  coincides with the null generators. In the region where it is timelike one may introduce the parameter  $\tau$  so that  $D^\mu \nabla_\mu \tau = 1$  and the metric takes the form

$$ds^2 = -d\tau^2 + \tau^2 q_{ij}(x^k) dx^i dx^j, \quad (25)$$

where  $q_{ij}$  is a positive definite  $n$ -metric.

### 3.3 Spacetime convex functions and submanifolds

Suppose a spacetime convex function  $f$  exists in  $(M, g_{\mu\nu})$ . Then, we deduce

**Proposition 7**  *$(M, g_{\mu\nu})$  admits no closed spacelike geodesics.*

*Proof.* The same argument as the Riemannian case.  $\square$

It is a non trivial question whether or not  $(M, g_{\mu\nu})$  admits a closed timelike geodesic. This will be discussed further in Section 6.

We also have

**Proposition 8** *Consider a spacetime  $(M, g_{\mu\nu})$  and a closed spacelike surface  $S \subset M$ . Suppose that in a neighbourhood of  $S$  the metric is written as*

$$g_{\mu\nu} dx^\mu dx^\nu = \gamma_{ab} dy^a dy^b + k_{pq} dz^p dz^q, \quad (26)$$

where  $k_{pq} dz^p dz^q$  is the metric on  $S$  and the components of the two-dimensional Lorentzian metric  $\gamma_{ab}$  are independent of the coordinates  $z^p$ . Then,  $S$  cannot be a closed marginally inner and outer trapped surface.

*Proof.* Let  $l^\alpha$  and  $n^\alpha$  satisfying  $l^\alpha l_\alpha = n^\alpha n_\alpha = 0$ ,  $l^\alpha n_\alpha = -1$  be the two null normals of a closed spacelike surface  $S$  so that  $\gamma_{ab} = -l_a n_b - n_a l_b$ . Then we have

$$\nabla^\mu \nabla_\mu f = {}^{(\gamma)}\nabla^a {}^{(\gamma)}\nabla_a f + {}^{(k)}\nabla^p {}^{(k)}\nabla_p f - (l^a n^b + n^a l^b) {}^{(\gamma)}\nabla_a \log \sqrt{k} {}^{(\gamma)}\nabla_b f, \quad (27)$$

where  ${}^{(\gamma)}\nabla_a$  denotes the Levi-Civita connection of  $\gamma_{ab}$  and  $k := \det(k_{pq})$ . Since  $(M, g_{\mu\nu})$  admits a spacetime convex function  $f$  and  $\partial_p \gamma_{ab} = 0$ , we have

$$\nabla_a \nabla_b f = {}^{(\gamma)}\nabla_a {}^{(\gamma)}\nabla_b f \geq c \gamma_{ab}. \quad (28)$$

Then from the partial trace

$$\nabla^a \nabla_a f = {}^{(\gamma)}\nabla^2 f \geq 2c, \quad (29)$$

and (27) we obtain

$${}^{(k)}\nabla^p {}^{(k)}\nabla_p f - (l^a n^b + n^a l^b) {}^{(\gamma)}\nabla_a \log \sqrt{k} {}^{(\gamma)}\nabla_b f = \nabla^\mu \nabla_\mu f - {}^{(\gamma)}\nabla^a {}^{(\gamma)}\nabla_a f \geq (n-1)c. \quad (30)$$

Since  $\sqrt{k}$  gives the area of  $S$ , if  $S$  is a closed marginally inner and outer trapped surface, it follows that

$$l^a(\gamma)\nabla_a \log \sqrt{k} = n^a(\gamma)\nabla_a \log \sqrt{k} = 0, \quad (31)$$

on  $S$ . Hence, it turns out from (30) that  $S$  admits a subharmonic function, i.e.,  ${}^{(k)}\nabla^2 f \geq (n-1)c > 0$ . This provides a contradiction.  $\square$

This suggests that the existence of a spacetime convex function might be incompatible with a worm hole like structure.

In Subsection 2.2, we pointed out that no marginally inner and outer trapped surface can exist on time symmetric initial hypersurface  $\Sigma$ , using a convex function  $\tilde{f}$  on a Riemannian submanifold  $\Sigma$  rather than a spacetime convex function  $f$ . We shall reconsider the previous situation. Suppose  $(M, g_{\mu\nu})$  contains an embedded spacelike hypersurface  $(\Sigma, h_{ij})$  with the second fundamental form  $K_{ij}$ . Then we have

$$\nabla_i \nabla_j f = D_i D_j f + K_{ij} \frac{\partial f}{\partial t}, \quad (32)$$

where  $\partial f / \partial t := t^\mu \partial_\mu f$  with  $t^\mu$  being a unit timelike vector,  $t^\mu t_\mu = -1$ , normal to  $\Sigma$ . Assuming that  $(M, g_{\mu\nu})$  admits a spacetime convex function  $f$ , we deduce

**Proposition 9** *If  $\Sigma$  is totally geodesic, i.e.,  $\Sigma$  is a surface of time symmetry, then  $(\Sigma, h_{ij})$  admits a convex function.*

*Proof.* This follows since  $K_{ij} = 0$ .  $\square$

Moreover we have

**Proposition 10** *If  $\Sigma$  is maximal, i.e.,  $h^{ij}K_{ij} = 0$ , it admits subharmonic function.*

**Corollary 1**  *$(\Sigma, h_{ij})$  cannot be closed.*

**Corollary 2**  *$(\Sigma, h_{ij})$  admits no closed geodesics nor minimal two surface in the case that  $K_{ij} = 0$ .*

As a simple example, consider the maximally extended Schwarzschild solution. In terms of the Kruskal coordinates  $(U, V) = (-e^{-(t-r_*)/4M}, e^{(t+r_*)/4M})$ ,  $r_* := r + 2M \log(r/2M - 1)$ , the metric is given by  $k_{pq} dz^p dz^q = r^2 d\Omega_{(2)}^2$  and  $\gamma_{ab} = -l_a n_b - n_a l_b$ , where

$$l_a = -F \partial_a V, \quad n_a = -F \partial_a U, \quad F^2 := \frac{16M^3 e^{-r/2M}}{r}. \quad (33)$$

Each  $U + V = \text{constant}$  hypersurface  $\Sigma$  is a time symmetric complete hypersurface which contains a marginally inner and outer trapped surface at  $U = V = 0$ . Hence, this spacetime cannot admit a spacetime convex function. This also accords with the claim in Subsection 2.2 that the complete Schwarzschild initial data set admits no strictly convex function.

## 4 Convex functions in cosmological spacetimes

Let us consider  $(n+1)$ -dimensional Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = \epsilon d\tau^2 + a(\tau)^2 q_{ij} dx^i dx^j, \quad (34)$$

where  $q_{ij} dx^i dx^j$  is a metric of  $n$ -dimensional space with constant curvature  $K = \pm 1, 0$ . We begin by considering functions which share the symmetry of the metric.



For an arbitrary function  $f$  of  $\tau$ , we have

$$\nabla_\tau \nabla_\tau f = \epsilon \ddot{f} g_{\tau\tau} , \quad (35)$$

$$\nabla_\tau \nabla_i f = 0 , \quad (36)$$

$$\nabla_i \nabla_j f = \epsilon \frac{\dot{a}}{a} \dot{f} g_{ij} , \quad (37)$$

where a dot denotes  $\tau$  derivative. In what follows we are concerned in particular with the case that  $\epsilon = -1$ ,  $q_{ij} dx^i dx^j$  is a Riemannian metric, and  $\dot{a} > 0$ : i.e., with an expanding universe.

Note the non-intuitive fact that what one usually thinks of as a “convex function of time”, i.e., one satisfying

$$\dot{f} > \text{constant} > 0 , \quad \ddot{f} > \text{constant}' > 0 , \quad (38)$$

is *never* a convex function in the classical sense because the RHS of (37) is negative. Moreover  $-f$  is *not* a concave function in the classical sense. However, the function  $-f$  *is* a spacetime convex function, provided  $\dot{a}$  is bounded below. This shows that if one wishes to use the idea of a convex function in cosmology one needs the spacetime definition.

We assume that the energy density  $\rho$  and the pressure  $p$  of matter contained in the universe obey  $p = w\rho$ . Then from the Einstein equations  $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$  and the conservation law  $\nabla_\nu T^\nu_\mu = 0$  we have

$$\dot{a}^2 = \sigma^2 a^{2-n(1+w)} - K , \quad (39)$$

where  $\sigma^2 := 2\kappa^2 \rho_0 / n(n-1)$  with  $\rho_0$  being the value of  $\rho$  at  $\tau_0$ . Naturally we require  $\rho \geq 0$ , so  $\sigma^2 \geq 0$ . Note that the RHS is positive and is bounded below by some positive constant in the  $K = -1$  case.

Now let us consider a function

$$f = -\frac{1}{2}a^2 . \quad (40)$$

If there exists a constant  $c > 0$  such that

$$\nabla_\tau \nabla_\tau f = -\dot{a}^2 - a\ddot{a} \geq -c , \quad (41)$$

$$\{\nabla_i \nabla_j f\} / g_{ij} = \dot{a}^2 \geq c . \quad (42)$$

then the function  $f$  is convex in the spacetime sense.

**(i)  $\ddot{a} = 0$ : uniformly expanding universe.**

From (41) and (42) we find that  $\dot{a}^2 = c$  and the function (40) is a spacetime convex function. In this case  $\nabla_\mu \nabla_\nu f = c g_{\mu\nu}$ , i.e.,  $\nabla_\mu f$  is a homothetic vector field, and the metric is the special case of (25).

From (39), it turns out that such a convex function is allowed on a vacuum open ( $\sigma^2 = 0$ ,  $K = -1$ ) FLRW universe, i.e., the Milne universe, and on flat and closed ( $K = 0$ ,  $+1$ ) FLRW universes with the matter satisfying

$$w = \frac{2-n}{n} , \quad \sigma^2 = c + K , \quad (43)$$

which is realized as, for example, a cosmic string dominated universe in the 4-dimensional case.

**(ii)  $\ddot{a} > 0$ : accelerately expanding universe.**

In this case (41) and (42) cannot hold simultaneously since  $a\ddot{a} > 0$ . Hence  $f$  cannot be a spacetime convex function. This implies that inflationary universes do not admit a convex function of the form (40).

**(iii)  $\ddot{a} < 0$ : decelerately expanding universe.**

From (41) and (42), if we can find a constant  $c > 0$  which satisfies the condition

$$0 \leq \dot{a}^2 - c \leq -a\ddot{a} , \quad -a\ddot{a} < \dot{a}^2 , \quad (44)$$

the function (40) becomes a spacetime convex function. Note that when the first condition holds but the second does not, the Hessian  $\nabla_\mu \nabla_\nu f$  becomes positive semi-definite and does not have Lorentzian signature. Note also that since (39) and its differentiation give

$$a\ddot{a} = \frac{2-n(1+w)}{2} \sigma^2 a^{2-n(1+w)} , \quad (45)$$

the assumption of deceleration  $\ddot{a} < 0$  requires that

$$w > \frac{2-n}{n}, \quad \sigma^2 > 0, \quad (46)$$

which corresponds to the timelike convergence condition,

$$R_{\tau\tau} = \kappa^2 \left( T_{\tau\tau} + \frac{1}{n-1} T_{\mu}^{\mu} \right) > 0. \quad (47)$$

In terms of  $w$  and  $\sigma^2$ , the condition (44) is written as

$$0 \leq \frac{\sigma^2}{a^{n(1+w)-2}} - K - c \leq \frac{n(1+w)-2}{2} \frac{\sigma^2}{a^{n(1+w)-2}} < \frac{\sigma^2}{a^{n(1+w)-2}} - K. \quad (48)$$

Considering a suitable open interval  $I \subset \mathbb{R}$ , we can find a constant  $c$  satisfying (48) for  $\tau \in I$ . Then  $f$  becomes a spacetime convex function on  $I \times \Sigma \subset M$ .

For example, when a radiation dominated open FLRW ( $n = 3, w = 1/3, K = -1$ ) universe is considered, the condition (48) is reduced to

$$1 \leq c \leq 1 + \frac{\sigma^2}{a^2}. \quad (49)$$

Then we can choose  $c = 1$  for  $I = (0, \infty)$  so that  $f$  becomes a spacetime convex function on  $I \times \Sigma$ , which covers the whole region of  $M$ . This can be, of course, verified directly by examining the solution  $a = \sigma \{(1 + \sigma^{-1}\tau)^2 - 1\}^{1/2}$  of (39).

In general,  $I \times \Sigma$  cannot be extended to cover the whole universe while keeping  $f$  a spacetime convex function. For example, in a closed ( $K = 1$ ) FLRW case the interval  $I$  should not contain the moment of maximal expansion, where  $\dot{a}$  vanishes. Indeed the timeslice of maximal expansion is a closed maximal hypersurface and according to Corollary 1 such an  $I \times \Sigma$  admits no spacetime convex functions.

In the following sections we shall discuss the issue of the global existence of convex functions, examining de-Sitter and anti-de-Sitter spacetime.

## 5 Convex functions and de-Sitter spacetime

In this section we shall consider convex functions in de-Sitter spacetime, which is a maximally symmetric spacetime with positive constant curvature.

### (i): Cosmological charts

We can introduce the three types of cosmological charts for which the metric takes the form of (34) with  $\epsilon = -1$  and  $q_{ij}dx^i dx^j$  being a Riemannian metric. The scale factors are given by

$$a(\tau) = \cosh \tau, \quad e^{\tau}, \quad \sinh \tau, \quad (50)$$

for  $K = 1, 0, -1$ , respectively. Note that the closed ( $K = 1$ ) FLRW chart covers the whole de-Sitter spacetime but the  $K = 0$  and  $K = -1$  FLRW charts do not.

In these cosmological charts, the function  $f$  of the form (40) cannot be a spacetime convex function. This agrees with the argument of the previous section since  $\ddot{a} > 0$  in the expanding region where  $\dot{a} > 0$ .

### (ii): Static chart

We can choose a coordinate system so that the metric has the static form,

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{(1-r^2)} + r^2 \Omega_{pq}^{(n-1)} dz^p dz^q. \quad (51)$$

The Hessian of the function

$$f = \frac{1}{2}r^2 \quad (52)$$

is given by

$$\nabla_t \nabla_t f = (1 - r^2)r^2, \quad \nabla_r \nabla_r f = (1 - 2r^2)g_{rr}, \quad \nabla_p \nabla_q f = (1 - r^2)g_{pq}. \quad (53)$$

Since  $\nabla_t \nabla_t f$  is positive in the region  $0 < r < 1$ , where  $g_{tt} < 0$ ,  $f$  cannot be a spacetime convex function, though it can become classically convex in the region  $0 < r < 1/2$ .

### (iii): Rindler chart

The de-Sitter metric can take the following form

$$ds^2 = d\tau^2 + \sin^2 \tau \left\{ -d\psi^2 + \cosh^2 \psi d\Omega_{(n-1)}^2 \right\}. \quad (54)$$

This corresponds to the metric (34) with  $\epsilon = +1$ ,  $q_{ij}dx^i dx^j$  being a Lorentzian metric of  $K = +1$ , and  $a(\tau) = \sin \tau$ . Let us consider the function

$$f = \frac{1}{2}a^2. \quad (55)$$

Then from (35) and (37), we can see that  $f$  would be a spacetime convex function if there was a constant  $c > 0$  such that  $c \leq \dot{a}^2 + a\ddot{a}$ ,  $\dot{a}^2 \leq c \leq \dot{a}^2$ . There is no such  $c$  since  $a\ddot{a} < 0$ , hence  $f$  is not spacetime convex.

## 6 Convex functions in anti-de-Sitter spacetime

We shall now examine the existence of convex function on anti-de-Sitter spacetime, which is maximally symmetric spacetime with negative constant curvature.

The metric of  $(n+1)$ -dimensional anti-de-Sitter spacetime may be given in the cosmological (open FLRW) form,

$$ds^2 = -d\tau^2 + \sin^2 \tau \left( d\xi^2 + \sinh^2 \xi d\Omega_{(n-1)}^2 \right), \quad (56)$$

where  $d\Omega_{(n-1)}^2$  is the metric of a unit  $(n-1)$ -sphere. This is the case that  $\epsilon = -1$ ,  $q_{ij}dx^i dx^j$  is the Riemannian metric with constant curvature  $K = -1$ , and  $a(\tau) = \sin \tau$  in (34). Since  $\dot{a} > 0$ ,  $\ddot{a} < 0$  for  $\tau \in (0, \pi/2)$ , according to the argument in Section 4, the function

$$f = -\frac{1}{2}a^2 = -\frac{1}{2}\sin^2 \tau \quad (57)$$

can be spacetime convex. Indeed, the Hessian is given by

$$\nabla_\tau \nabla_\tau f = -\cos 2\tau, \quad \nabla_\tau \nabla_i f = 0, \quad \nabla_i \nabla_j f = \cos^2 \tau g_{ij}, \quad (58)$$

hence (57) is spacetime convex at least locally. If the time interval  $I$  contains a moment  $\tau = \pi/2$ , then (57) is no longer strictly convex in such a  $I \times \Sigma$ .

Anti-de-Sitter spacetime admits closed timelike geodesics. Therefore the existence of a locally spacetime convex function does not exclude closed timelike geodesics.

The cosmological chart (56) does not cover the whole spacetime; it covers the region I as illustrated in the figure 2 and the boundary is a Cauchy horizon for  $\tau = \text{constant}$  hypersurfaces. To see whether or not the locally spacetime convex function  $f$  discussed above is extendible beyond the horizon while maintaining its spacetime convexity, let us consider the embedding of  $(n+1)$ -dimensional anti-de-Sitter spacetime in  $\mathbb{E}^{n,2}$  as a hyperboloid:

$$-(X^0)^2 - (X^1)^2 + (X^1)^2 + \cdots + (X^{n+1})^2 = -1, \quad (59)$$

where  $X^a$  are the Cartesian coordinates of  $\mathbb{E}^{n,2}$ . The open FLRW chart (56) is then introduced by

$$X^0 = \cos \tau, \quad X^i = \sin \tau z^i, \quad (i = 1, \dots, n+1), \quad (60)$$

where  $z^i \in \mathbb{H}^n$ , i.e.,  $-(z^1)^2 + (z^2)^2 + \cdots + (z^{n+1})^2 = -1$ . One can observe that  $|X^0| < 1$  in region I,  $X^0 = 1$  at the horizon, and  $1 < |X^0|$  beyond the horizon (region II).

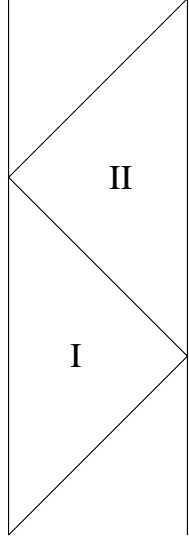


Figure 2: The Penrose diagram of the universal covering space of anti-de-Sitter spacetime. A single and double vertical lines denote a centre of the spherical symmetry and the infinity, respectively.

In terms of the embedding coordinates, the locally convex function (57) is expressed as

$$f = -\frac{1}{2} \{1 - (X^0)^2\} . \quad (61)$$

This expression is not restricted to  $|X^0| < 1$  and hence can be regarded as an extension of (57) beyond the horizon.

Now, let us consider a function  $\phi = X^0$ . We see that

$$\xi^\mu := \nabla^\mu \phi , \quad (62)$$

is a conformal Killing vector field, i.e.,  $\nabla_\mu \xi_\nu = \nabla_\mu \nabla_\nu \phi = \sigma g_{\mu\nu}$ , and, moreover,  $\sigma = \phi$ . Indeed, in the region I where  $\phi = \cos \tau$ , we can directly verify the equation

$$\nabla_\mu \nabla_\nu \phi = \phi g_{\mu\nu} . \quad (63)$$

Since this expression is covariant and  $\phi$  is globally defined, this equation holds globally. Note that  $\xi^\mu$  becomes timelike in region I, null at the horizon, and spacelike in region II, since its norm is given by  $\xi^\mu \xi_\mu = -1 + \phi^2$ .

Now, rewriting (61) as  $f = -(1 - \phi^2)/2$ , we obtain

$$\nabla_\mu \nabla_\nu f = \phi \nabla_\mu \nabla_\nu \phi + \nabla_\mu \phi \nabla_\nu \phi = \phi^2 g_{\mu\nu} + \xi_\mu \xi_\nu . \quad (64)$$

This is the global expression for the Hessian of  $f$ . In region II,  $\xi^\mu$  becomes spacelike and there is a unit timelike vector field  $t^\mu$  which is orthogonal to  $\xi^\mu$ , i.e.,  $t^\mu \xi_\mu = 0$ . Then, contraction of (64) with  $t^\mu$  gives  $t^\mu t^\nu \nabla_\mu \nabla_\nu f = -\phi^2$ . Since  $\phi^2 = (X^0)^2$  is not bounded above in region II, for any fixed  $c > 0$ ,  $t^\mu t^\nu \nabla_\mu \nabla_\nu f \geq c g_{\mu\nu} t^\mu t^\nu$  fails in the region where  $\phi^2 > c$ . Thus, the function (61) cannot be globally spacetime convex.

## 7 Convex functions and black hole spacetimes

Let us consider  $(n+1)$ -dimensional static metric;

$$ds^2 = \epsilon \Delta dt^2 + \frac{dr^2}{\Delta} + R^2(r) d\Omega_{(n-1)}^2 , \quad (65)$$

where  $\epsilon = \pm 1$ ,  $\Delta$  is a function of  $r$ , and  $d\Omega_{(n-1)}^2 = \Omega_{pq} dz^p dz^q$  is the metric of  $(n-1)$ -dimensional space with unit constant curvature  $K$ .

For an arbitrary function  $f = f(t, r)$ , we have

$$\nabla_t \nabla_t f = \left( \frac{\epsilon}{\Delta} \ddot{f} + \frac{\Delta'}{2} f' \right) g_{tt} , \quad (66)$$

$$\nabla_r \nabla_r f = \left( \Delta f'' + \frac{\Delta'}{2} f' \right) g_{rr} , \quad (67)$$

$$\nabla_p \nabla_q f = \Delta \frac{R'}{R} f' g_{pq} , \quad (68)$$

where the *dot* and the *prime* denote  $t$  and  $r$  derivatives, respectively.

Consider a function

$$f = \frac{1}{2} (R^2 - \alpha t^2) , \quad (69)$$

outside the event horizon, where  $\Delta > 0$ . This function asymptotically approaches the canonical one (22). We see from (66), (67), and (68) that  $f$  can be a spacetime convex function if there exists a constant  $c > 0$  satisfying

$$0 < \frac{\alpha}{\Delta} + \frac{\Delta'}{2} R R' \leq c , \quad \Delta(R'^2 + R R'') + \frac{\Delta'}{2} R R' \geq c , \quad \Delta R'^2 \geq c . \quad (70)$$

Let us examine the function (69) on the  $(n+1)$ -dimensional ( $n \geq 3$ ) Schwarzschild spacetime, whose metric is given by

$$\epsilon = -1 , \quad \Delta = 1 - \left( \frac{r_g}{r} \right)^{n-2} , \quad R = r , \quad (71)$$

where  $r_g$  is the radius of the event horizon. Since  $\Delta$  and  $\alpha/\Delta + \Delta' r/2$  are monotonically increasing and decreasing function, respectively, one can observe that, for  $0 \leq \alpha < 1$ , there exists a radius  $r_*$  ( $> r_g$ ):

$$r_*^{n-2} = \frac{n+2 + \sqrt{(n+2)^2 - 8n(1-\alpha)}}{4(1-\alpha)} r_g^{n-2} , \quad (72)$$

for which

$$\Delta(r_*) = \frac{\alpha}{\Delta(r_*)} + \frac{\Delta'(r_*)}{2} r_* . \quad (73)$$

Then, if one chooses

$$c = \Delta(r_*) , \quad (74)$$

condition (70) holds in the region  $r_* \leq r < \infty$ , hence  $f$  becomes a spacetime convex function there.

Note that, in the black hole spacetime, the Hessian of (69) has Lorentzian signature even for  $\alpha = 0$ , while in the strictly flat spacetime, the Hessian becomes positive semi-definite for  $\alpha = 0$ .

One can also see that, for  $\alpha < 0$ , the condition (70) cannot hold in the asymptotic region, and for  $1 \leq \alpha$ , (70) holds nowhere.

If one takes  $\Delta$  in (71) as

$$\Delta = 1 + \left( \frac{r_g}{r} \right)^{n-2} , \quad (75)$$

then the metric (65) describes the negative mass Schwarzschild spacetime, which has a timelike singularity and no horizon.

One can observe that the function  $\{\nabla_t \nabla_t f\}/g_{tt}$  is a monotonically increasing function with the range  $(-\infty, \alpha)$ , and  $\{\nabla_p \nabla_q f\}/g_{pq}$  monotonically decreasing function with the range  $(1, \infty)$ . On the other hand, the function  $\{\nabla_r \nabla_r f\}/g_{rr}$  is a decreasing function with the range  $(1, \infty)$  in  $n = 3$  case and constant  $= 1$  in  $n = 4$  case, while it becomes monotonically increasing with the range  $(-\infty, 1)$  in the case  $n \geq 5$ .

Therefore, in the cases  $n = 3, 4$ , if  $\alpha \leq c \leq 1$ , then the function (69) with  $(0 < \alpha \leq 1)$  is spacetime convex in the region  $\hat{r}_* < r < \infty$ , where  $\hat{r}_*$  is given by  $\{\nabla_t \nabla_t f\}/g_{tt}(\hat{r}_*) = 0$ . In the case  $n \geq 5$ , if one chooses  $\alpha \leq c < 1$ , then the function (69) with  $(0 < \alpha < 1)$  is a spacetime convex function in the region  $\max\{\hat{r}_*, \hat{r}_*\} < r < \infty$ , where  $\hat{r}_*$  is given by  $\{\nabla_r \nabla_r f\}/g_{rr}(\hat{r}_*) = c$ . Thus we were not able to find a spacetime convex function through the region  $r > 0$  of the negative mass Schwarzschild solution.

## 8 Level sets and foliations

Locally the level sets  $\Sigma_c = \{x \in M | f(x) = \text{constant} = c\}$  of a function  $f$  have a unit normal given by

$$n_\mu = -\frac{\nabla_\mu f}{\sqrt{\epsilon \nabla_\nu f \nabla^\nu f}}, \quad (76)$$

where  $\epsilon = -1$  if the normal is timelike and  $\epsilon = 1$  if it is spacelike. Given two vectors  $X^\mu$  and  $Y^\nu$  tangent to  $\Sigma_c$  so that  $X^\mu \partial_\mu f = 0 = Y^\nu \partial_\nu f$  one may evaluate  $K_{\mu\nu} X^\mu Y^\nu$  in terms of the Hessian of  $f$ :

$$K_{\mu\nu} X^\mu Y^\nu = X^\mu Y^\nu \frac{\nabla_\mu \nabla_\nu f}{\sqrt{\epsilon \nabla_\nu f \nabla^\nu f}}. \quad (77)$$

Thus, in the Riemannian case (for which the metric  $g_{\mu\nu}$  on  $M$  is positive definite and  $\epsilon = 1$ ), a strictly convex function has a positive definite second fundamental form. Of course there is a convention here about the choice of direction of the normal. We have chosen  $f$  to decrease along  $n^\mu$ . The converse is not necessarily true, since  $f(x)$  and  $g(f(x))$  have the same level sets, where  $g$  is a monotonic function of  $\mathbb{R}$ . Using this gauge freedom we may easily change the signature of the Hessian. However, given a hypersurface  $\Sigma_0$  with positive definite second fundamental form, we may, locally, also use this gauge freedom to find a convex function whose level  $f = 0$  coincides with  $\Sigma_0$ . If we have a foliation (often called a “slicing” by relativists) by hypersurfaces with positive definite fundamental form, we may locally represent the leaves as the levels sets of a convex function.

Similar remarks apply for Lorentzian metrics. The case of greatest interest is when the level sets have a timelike normal. For a classical strictly convex function, the second fundamental form will be positive definite and the hypersurface orthogonal timelike congruence given by the normals  $n^\mu$  is an expanding one. For a spacetime convex function, our conventions also imply that the second fundamental form of a spacelike level set is positive definite. This can be illustrated by the canonical example (22) with  $\alpha = 1$  in flat spacetime. The spacelike level sets foliate the interior of the future (or past) light cone. Each leaf is isometric to hyperbolic space. The expansion is homogeneous and isotropic, and if we introduce coordinates adapted to the foliation we obtain the flat metric in  $K = -1$  FLRW form (34) with scale factor  $a(\tau) = \tau$ . This is often called the Milne model. The convex function underlying this FLRW coordinate system is globally well defined but the coordinate system based on it breaks down at the origin of Minkowski spacetime.

From what has been said it is clear that there is a close relation between the existence of convex functions and the existence of foliations with positive definite second fundamental form. A special case, often encountered in practice, are totally umbilic foliations. These have  $K_{ij} = \frac{1}{n} g_{ij} \text{Tr} K$ . In a 4-dimensional Einstein space for which  $R_{\mu\nu} = \Lambda g_{\mu\nu}$  the Gauss-Codazzi equations imply that 3-dimensional umbilic hypersurface  $\Sigma_c$  is a constant curvature space with a sectional curvature  $C$  and the trace of  $K_{ij}$  is given by  $\text{Tr} K = 3\sqrt{\frac{\Lambda}{3}} - C$ . The sectional curvature  $C$  could depend only upon “time function”  $f$  and thus totally umbilic hypersurfaces are a special case of “trace  $K = \text{constant}$ ” hypersurfaces which are used in numerical relativity. One of the simplest cases is given by considering in Minkowski space, in which trace  $K$  constant, often referred to as “constant mean curvature” hypersurfaces are hyperbolic and convex (see e.g. [T] and references therein). The Milne slicing mentioned above is such an obvious example. In the case of de-Sitter or anti-de-Sitter spacetimes, which may be represented by quadrics in five dimensions, one may obtain totally umbilic hypersurfaces by intersecting the quadric with a hyperplane. Many standard coordinate systems for de-Sitter and anti-de-Sitter spacetime are obtained in this way. The singularities of these coordinate systems may sometimes be seen in terms of the non-existence of convex functions.

## 9 Barriers and Convex functions

In the theory of maximal hypersurfaces and more generally hypersurfaces of constant mean curvature (“ $\text{Tr}K = \text{constant}$ ” hypersurface) an important role is played by the idea of a “barriers.” The basic idea is that, if two spacelike hypersurfaces  $\Sigma_1$  and  $\Sigma_2$  touch, then the one in the future,  $\Sigma_2$  say, can have no smaller a mean curvature than the  $\Sigma_1$ , the hypersurface in the past, i.e.,  $\text{Tr}K_1 \leq \text{Tr}K_2$ . This is illustrated in the accompanying figure 3:

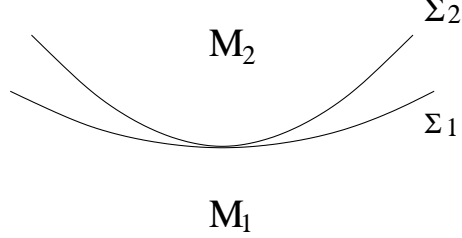


Figure 3: Spacelike hypersurfaces,  $\Sigma_1$  and  $\Sigma_2$ .

More formally, we have, using the Maximum Principle, the following theorem:

**Theorem 1 Eschenburg [E]:** *Let  $M_1$  and  $M_2$  be disjoint open domains with spacelike connected  $C^2$ -boundaries having a point in common. If the mean curvatures  $\text{Tr}K_1$  of  $\partial M_1$  and  $\text{Tr}K_2$  of  $\partial M_2$  satisfy*

$$\text{Tr}K_1 \leq -a, \quad \text{Tr}K_2 \leq a, \quad (78)$$

*for some real number  $a$ , then  $\partial M_1 = \partial M_2$ , and  $\text{Tr}K_2 = -\text{Tr}K_1 = a$ .*

As an example, let us suppose that  $\Sigma_1$  is maximal, i.e.,  $\text{Tr}K_1 = 0$ , where  $K_1$  is the second fundamental form of  $\Sigma_1$ . Now consider a function  $f$  whose level sets lie in the future of  $\Sigma_1$  and one of which touches  $\Sigma_1$  as in the figure 4.

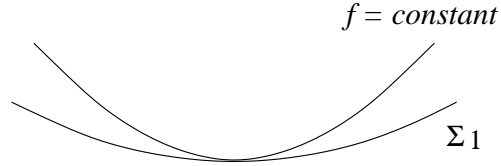


Figure 4: One of the level sets of  $f$  and a maximal hypersurface  $\Sigma_1$ .

The second fundamental form  $K_f$  of the level set  $f = \text{constant}$  must have positive trace,  $\text{Tr}K_f > 0$ . In particular,  $\text{Tr}K_f < 0$  is excluded. Now if  $f$  is a spacetime *concave* function, i.e.,

$$\nabla_\mu \nabla_\nu f < -c g_{\mu\nu}, \quad (79)$$

for  $c > 0$ , we have  $\text{Tr}K_f < 0$ . It follows that the level sets of a concave function cannot penetrate the “barrier” produced by the maximal hypersurface  $\Sigma_1$ .

As an application of this idea, consider the interior region of the Schwarzschild solution. The hypersurface  $r = \text{constant}$  has

$$\text{Tr}K(r) = -\frac{2}{r} \left( \frac{2M}{r} - 1 \right)^{-1/2} \left( 1 - \frac{3M}{2r} \right). \quad (80)$$

The sign of  $\text{Tr}K(r)$  is determined by the fact that regarding  $r$  as a time coordinate,  $r$  decrease as time increases. Thus for  $r > 3M/2$ ,  $\text{Tr}K$  is negative but for  $r < 3M/2$ ,  $\text{Tr}K$  is positive. If  $r = 3M/2$ , we have  $\text{Tr}K = 0$ , that is,  $r = 3M/2$  is a maximal hypersurface.

Consider now attempting to foliate the interior region II by the level sets of a concave function. Every level set must touch an  $r = \text{constant}$  hypersurface at some point. If  $r = 3M/2$  this can, by the theorem, only happen if the level set lies in the past of the hypersurface  $r = 3M/2$  as depicted in the figure 5. Evidently therefore a foliation by level sets of a spacetime *concave* function can never penetrate the

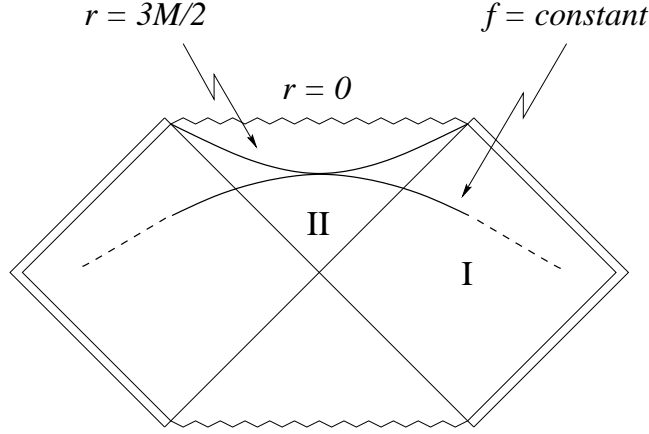


Figure 5: Penrose diagram of the Schwarzschild black hole. Level sets of a concave function  $f$  must be in the past of the barrier hypersurface  $r = 3M/2$ .

barrier at  $r = 3M/2$ . In particular, such a foliation can never extend to the singularity at  $r = 0$ .

These results are relevant to work in numerical relativity. One typically sets up a coordinate system in which the constant time surfaces are of constant mean curvature, i.e., constant trace  $K$ . It follows from our results that, if the constant is *negative*, then, the coordinate system can never penetrate the region  $r < 3M/2$ . In fact it could never penetrate any maximal hypersurface.

Now we shall illustrate the remarks mentioned above with an example of a constant mean curvature foliation in a black hole spacetime. We are concerned with a spacelike hypersurface  $\Sigma \subset M$  whose mean curvature is constant  $\lambda$  (or zero). A convenient way to find such hypersurfaces in a given spacetime is to use the variational principle. In this approach, the behaviours of constant mean curvature slices in the Schwarzschild black hole have been studied by Brill *et.al* [BCI].

Let us start from an action

$$S = \int_{\Sigma} d^n x \sqrt{h} + \lambda \int_M d^{n+1} x \sqrt{-g}, \quad (81)$$

where  $h$  and  $g$  denote the determinant of the metric  $h_{ij}$  on  $\Sigma$  and that of a spacetime metric  $g_{\mu\nu}$ , respectively. In other words, the first term is the volume of  $\Sigma$  and the second integral represents a spacetime volume enclosed by  $\Sigma$  together with any fixed hypersurface. The variation of this action gives

$$n_{\mu} h^{ij} (D_i \partial_j x^{\mu} + \Gamma^{\mu}_{\nu\sigma} \partial_i x^{\nu} \partial_j x^{\sigma}) = \lambda, \quad (82)$$

where  $n^{\mu}$  is a unit normal vector to  $\Sigma$  and  $\Gamma^{\mu}_{\nu\lambda}$  is the Christoffel symbol of  $g_{\mu\nu}$ . Using the Gauss-Weingarten equation, one can see that (82) is equivalent to

$$\text{Tr} K = \lambda. \quad (83)$$

Thus, the solutions of the differential equation (82) describe constant mean curvature hypersurfaces  $\Sigma$ .

Consider now an  $(n+1)$ -dimensional black hole metric (65),

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = -\Delta dt^2 + \frac{dr^2}{\Delta} + R^2(r) \Omega_{pq} dz^p dz^q, \quad (84)$$



and look for  $\Sigma$ , which we assume to be spherically symmetric, for simplicity. As we will see, the first integral of (82) is easily obtained because the metric (84) has a Killing symmetry.

Let us choose the coordinates  $x^i$  on  $\Sigma$  as  $(t, z^p)$ , then  $\Sigma$  is described by  $r = r(t)$  and the induced metric is written as

$$h_{ij}dx^i dx^j = (-\Delta + \Delta^{-1}\dot{r}^2) dt^2 + R^2(r) d\Omega_{(n-1)}^2, \quad (85)$$

where the *dot* denotes the derivative by  $t$ . Then, the action (81) takes the form,

$$S = \int dt L = \int dt \left\{ R^{n-1} \sqrt{-\Delta + \Delta^{-1}\dot{r}^2} - \lambda F \right\}, \quad F(r) := \int^r d\bar{r} R^{n-1}(\bar{r}), \quad (86)$$

where the integration of angular part is omitted. The coordinate  $t$  is cyclic, hence the Hamiltonian,

$$H = \frac{\partial L}{\partial \dot{r}} \dot{r} - L = \frac{R^{n-1} \Delta}{\sqrt{-\Delta + \Delta^{-1}\dot{r}^2}}, \quad (87)$$

is conserved:  $H = \text{constant} =: E$ . Then, from (87), one obtains a first integral,

$$\left( \frac{1}{\Delta} \frac{dr}{dt} \right)^2 - \frac{\Delta R^{2(n-1)}}{(E - \lambda F)^2} = 1, \quad (88)$$

which is analogous to the *law of energy conservation* for a point particle moving in an effective potential,

$$V_{\text{eff}} := - \frac{\Delta R^{2(n-1)}}{(E - \lambda F)^2}. \quad (89)$$

The geometry of  $\Sigma$  is determined, once  $(E, \lambda)$ , and an “initial” point are given. The first and second fundamental forms of  $\Sigma$  are written, in the coordinates  $(r, z^p)$ , as

$$h_{ij}dx^i dx^j = \frac{1}{\Delta + \mu^2} dr^2 + R^2 \Omega_{pq} dz^p dz^q, \quad (90)$$

$$K_r^r = \lambda + (n-1)\mu \frac{R'}{R}, \quad (91)$$

$$K_q^p = -\mu \frac{R'}{R} \delta_q^p, \quad (92)$$

where  $\mu(r) := \{E - \lambda F(r)\} R^{-(n-1)}$ .

The qualitative behaviours of  $\Sigma$  in  $(M, g_{\mu\nu})$  can be understood by observing the shape of the function  $V_{\text{eff}}$ . Inside the black hole, where  $\Delta < 0$ ,  $V_{\text{eff}}$  is positive and the extremal of  $V_{\text{eff}}$  provides a  $r = \text{constant}$  hypersurface. Actually, for example in the Schwarzschild spacetime, one can obtain a  $r = \text{constant}$  hypersurface for any  $r(= R) < r_g$ , by choosing  $(E, \lambda)$  such that

$$E^2 = \frac{r^n (2r^{n-2} - nr_g^{n-2})^2}{4n^2 (r_g^{n-2} - r^{n-2})}, \quad \lambda^2 = \frac{\{2(n-1)r^{n-1} - nr_g^{n-2}\}^2}{4r^n (r_g^{n-2} - r^{n-2})}, \quad (93)$$

for which  $V_{\text{eff}} = 1$  and  $dV_{\text{eff}}/dr = 0$  are satisfied.

On the other hand, outside the event horizon,  $V_{\text{eff}}$  is negative and there is no  $r = \text{constant}$  hypersurface. If  $\lambda \neq 0$ ,  $V_{\text{eff}} \rightarrow 0$  as  $r \rightarrow \infty$ , and then (88) means  $dr/dt \rightarrow 1$ , that is,  $\Sigma$  turns out to be asymptotically null and goes to null infinity. If  $\lambda = 0$ , i.e.,  $\Sigma$  is maximal, then  $\Sigma$  goes to spacelike infinity.

Note that the solutions of (88) do not involve  $t = \text{constant}$  hypersurfaces. Such a hypersurface is time-symmetric, and thus totally geodesic,  $K_{ij} = 0$ .

The numerical integration of (88) has been done by Brill *et.al* [BCI] in the 4-dimensional ( $n = 3$ ) Schwarzschild black hole case, in which the effective potential becomes

$$V_{\text{eff}} = - \frac{r^3(r - 2M)}{\left(E - \frac{\lambda}{3}r^3\right)^2}. \quad (94)$$

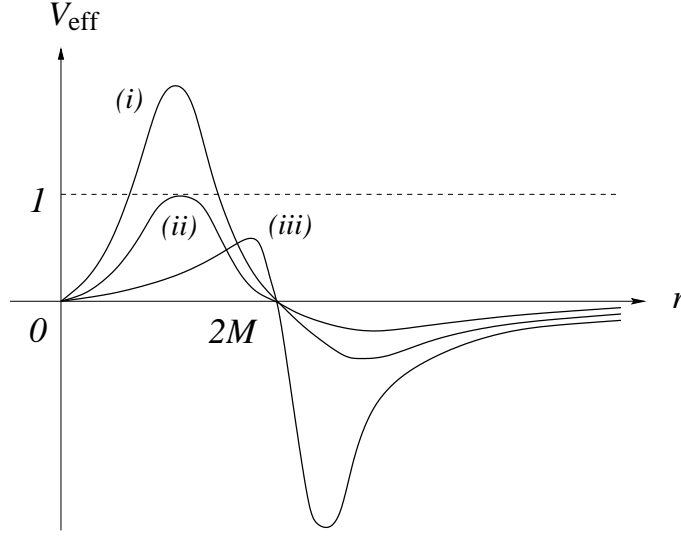


Figure 6: The effective potential  $V_{\text{eff}}$  of  $\lambda \neq 0$  case, which becomes zero at  $r = 0$  and  $2M$  and  $\sim -r^{-2}$  as  $r \rightarrow \infty$ . (i): There are turning points.  $\Sigma$  has either two singularities, or none at all. (ii): There is an unstable equilibria point.  $\Sigma$  is a  $r = \text{constant}$  hypersurface. (iii): There is no turning point.  $\Sigma$  contains one and only one singularity.

The shape of the function  $V_{\text{eff}}$  and some typical hypersurfaces are schematically depicted in the figures 6, and 7, respectively [BCI].

Some of the hypersurfaces which do not have a *turning point*, where  $V_{\text{eff}} = 1$  or  $dr/(\Delta dt) = 0$ , necessarily hit the singularity at  $r = 0$ . It has been shown that for each fixed value of  $\lambda$ , there exit values  $E_+$  and  $E_-$  such that all hypersurfaces with  $E < E_-$  or  $E > E_+$  contain one and only one singularity, while those with  $E_- < E < E_+$  contain either two singularities or none at all. The value  $E_{\pm}$  are determined by (93) and the hypersurfaces  $E = E_+$  or  $E = E_-$  are the homogeneous  $r = \text{constant}$  hypersurfaces, which are the barriers for the regular hypersurfaces with a given  $\lambda$ . For example, in the maximal case  $\lambda = 0$ , the barrier hypersurface is given by  $r = 3M/2$  and any regular maximal hypersurfaces do not exist inside this hypersurface.

The point is that if  $\lambda$  is large enough, regular constant mean curvature hypersurfaces can approach the singularity at  $r = 0$  arbitrarily closely. Thus, as discussed before, the regular  $\text{Tr}K = \lambda > 0$  hypersurfaces can penetrate the barrier hypersurface  $r = 3M/2$ . However a foliation by  $\text{Tr}K < 0$  hypersurfaces, on each one of which  $r$  attains a minimum value cannot penetrate the barrier.

We should comment that  $\text{Tr}K > 0$  does not always mean that  $\Sigma$  is a convex hypersurface. Actually, in the present example, one of the components of the second fundamental form (91) and (92) can be negative even when  $\text{Tr}K > 0$ . The existence of a spacetime convex function is thus a sufficient condition but not a necessary condition for the existence of a convex hypersurface.

A particularly interesting case is that of  $E = 0$  with  $r = R$ , for which the second fundamental form becomes

$$K_{ij} = \frac{\lambda}{n} h_{ij} , \quad (95)$$

hence  $\Sigma$  is a totally umbilic convex hypersurface. In this case, as discussed in the previous section 8, one may find a convex function whose level coincides with  $\Sigma$ .

We also remark that  $\Sigma$  which has a turning point contains a minimal surface  $S \subset \Sigma$ . The second fundamental form  $H_{pq}$  of  $r = \text{constant}$  sphere  $S \subset \Sigma$  is given by

$$H_{pq} = -\sqrt{\mu^2(1 - V_{\text{eff}})} \frac{R'}{R} h_{pq} . \quad (96)$$

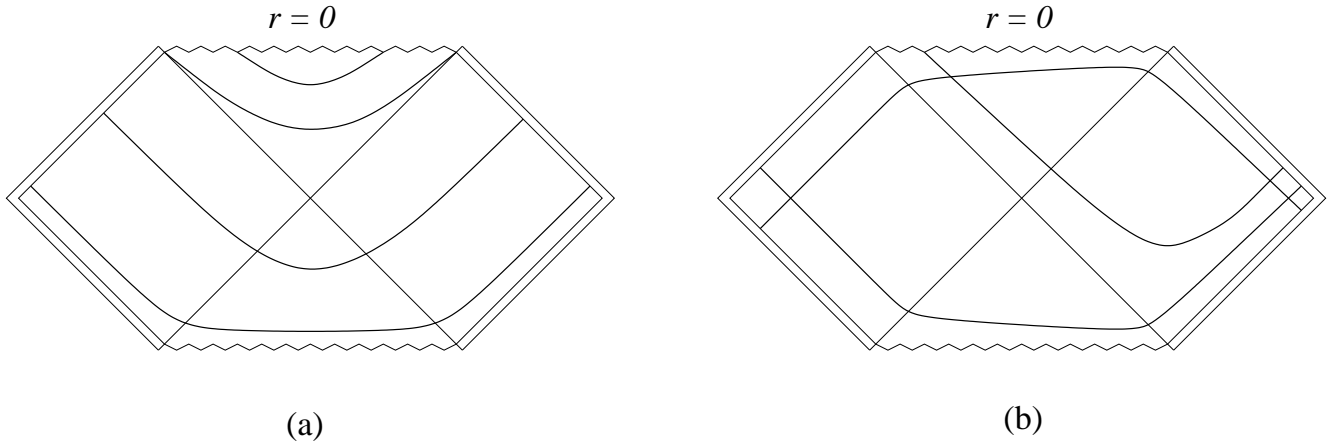


Figure 7: Penrose diagram of the Schwarzschild black hole and a variety of constant mean curvature hypersurfaces with various  $(E, \lambda)$ . (a) Hypersurfaces with reflection symmetry. (b) No-reflection symmetric hypersurfaces.

Thus  $H_{pq} = 0$  at the turning point. Hence  $S$  is minimal there. This also means that  $(\Sigma, h_{ij})$  contains closed geodesics on  $S$ . Since all regular hypersurfaces have a turning point, it turns out from Proposition 3 that

**Remark 1** *Regular constant mean curvature hypersurfaces in a black hole spacetime do not admit a strictly or uniformly convex function which lives on the hypersurface.*

However, regular  $\Sigma$ s which foliate the interior of a black hole do not intersect the closed marginally inner and outer trapped surface. Thus, there remains a possibility that  $(M, g_{\mu\nu})$  admits a *spacetime convex function* whose level sets coincide with  $\Sigma$ s.

The existence problem of constant mean curvature foliations has been investigated extensively not only in black hole spacetimes as discussed here but also in cosmological spacetimes, or spacetimes having a compact Cauchy surface (see, e.g. [Br, Gr, IR]). In cosmological spacetimes, constant mean curvature hypersurfaces, if exist, are likely to be compact, and thus do not admit a strictly or uniformly convex function which live on the hypersurfaces, because of Corollary 1. However, a *spacetime convex function*, if available, can give a constant mean curvature foliation with non-vanishing mean curvature as its level surfaces.

## 10 Conclusion

We have discussed consequences of the existence of convex functions on spacetimes. We first considered convex functions on a Riemannian manifold. We have shown that the existence of a suitably defined convex function is incompatible with closedness of the manifold and the existence of a closed minimal submanifold. We also discussed the relation to Killing symmetry and homothety. Next we gave the definition of a convex function on a Lorentzian manifold so that its Hessian has Lorentzian signature and the light cone defined by the Hessian lies inside the light cone defined by metric. Spacetimes admitting spacetime convex functions thus have a particular type of causal structure. We have shown that spacetime on which a spacetime convex function exists does not admit a closed marginally inner and outer trapped surface. We gave examples of spacetime convex functions on cosmological spacetimes, anti-de-Sitter spacetime, and a black hole spacetime. Level sets of a convex function provide convex hypersurfaces. We have discussed level sets of convex functions, barriers, and foliations by constant mean curvature hypersurfaces. We anticipate that further study of convex functions and foliations by convex surfaces will provide additional insights into global problems in general relativity and should have applications to numerical relativity.

## Acknowledgments

G.W.G. would like to thank Profs. T. Nakamura and H. Kodama for their kind hospitality at the Yukawa Institute for Theoretical Physics where the main part of this work was done. A.I. would like to thank members of DAMTP for their kind hospitality. A.I. was supported by Japan Society for the Promotion of Science.

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